Q. Let S be the set of all real valued continuous function defined on the closed interval $[0,1]$. Define addition and multiplication on S by

$(f+g)(x) = f(x) + g(x)$, $x \in [0,1]$ for $f, g \in S$.

$(f \cdot g)(x) = f(x) \cdot g(x)$, $x \in [0,1]$ for $f, g \in S$.

Prove that $(S, +, \cdot)$ is a commutative ring with unity. Show that the ring contains divisor of zero.

Ans: Clearly, the set S is nonempty. Also since the sum and product of two ~~of~~ ~~two~~ real valued continuous function are again continuous functions, we have. S is closed on $[0,1]$ and '$+$' and '$\cdot$'

Now for any $x \in [0,1]$ and any $f, g, h \in S$, we have

(i) $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.

$\Rightarrow f+g = g+f$.

(ii) $((f+g)+h)(x) = (f+g)(x) + h(x)$

$= (f(x) + g(x)) + h(x)$

$= f(x) + g(x) + h(x)$

$= f(x) + (g(x) + h(x))$

$= f(x) + (g+h)(x)$

$= (f + (g+h))(x)$

$\therefore ((f+g)+h) = (f + (g+h))$

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(iii) The constant function 0, which sends all the elements of $[0,1]$ to zero, satisfies $(f+0)(x) = f(x)+0(x)$
$= f(x), \forall f \in S$

$\therefore f+0 = f.$

(iv) For each $f \in S$, define a function $f : S \to [0,1]$ by

$(-f)(x) = -f(x).$

Then $(f+(-f))(x) = f(x)+(-f(x)) = f(x)-f(x) = 0 = 0(x).$

i.e. $f+(-f) = 0$

(v) $((f \cdot g) \cdot h)(x) = (f \cdot g)(x) \cdot h(x) = (f(x) \cdot g(x))\, h(x)$

$= f(x)\,(g(x) \cdot h(x))$

$= f(x)\,(g \cdot h)(x).$

$= (f \cdot (g \cdot h))(x).$

Thus, $((f \cdot g) \cdot h) = (f \cdot (g \cdot h))$

(vi) $(f \cdot (g+h))(x) = f(x) \cdot (g+h)(x).$

$= f(x) \cdot (g(x)+h(x))$

$= f(x)g(x) + f(x)h(x)$

$= (f \cdot g)(x) + (f \cdot h)(x).$

$= ((f \cdot g)+(f \cdot h))(x).$

$\therefore (f \cdot (g+h)) = (f \cdot g) + (f \cdot h)$

Similarly, $(g+h) \cdot f = g \cdot f + h \cdot f.$

$\therefore (S, +, \cdot)$ is a commutative ring.

Now, $(f \cdot 1)(x) = f(x)$

and $(1 \cdot f)(x) = f(x).$

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$\therefore$ 1 is the unity element.

Since the constant function is continuous.

So, $1 \in S$.

Hence, $(S, +, \cdot)$ is a commutative ring with unity.

Now, let, $f(x) = \begin{cases} 0, & x \in [0, 1/2] \\ x - 1/2, & x \in [1/2, 1] \end{cases}$

and $g(x) = \begin{cases} x - 1/2, & x \in [0, 1/2] \\ 0, & x \in [1/2, 1] \end{cases}$

Then $f \neq 0$, $g \neq 0$

$f \cdot g = 0$

$\therefore$ The ring contains the divisor of zero.